AD-A015 703

ON THE STAPILITY OF NON-AUTONOMOUS DIFFERENTIAL EQUATIONS $dx/dt=[A + B(t)]x$, WITH SKEW-SYMMETRIC MATRIX $B(t)$

A. P. Morgan, et al

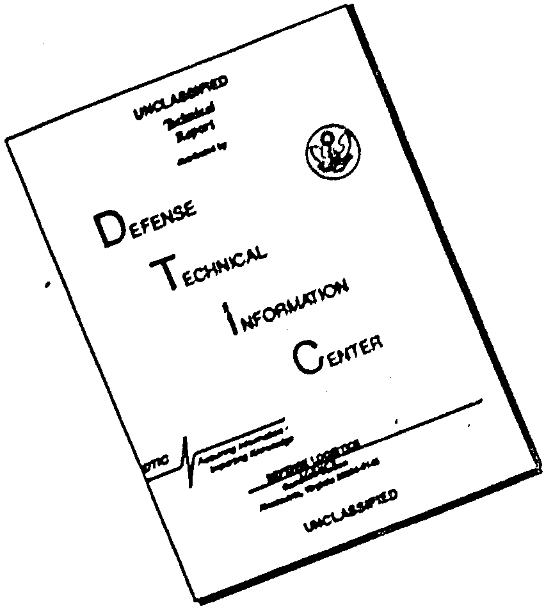
Yale University

Prepared for:

Office of Naval Research

August 1975

DISCLAIMER NOTICE



THIS DOCUMENT IS BEST QUALITY AVAILABLE. THE COPY FURNISHED TO DTIC CONTAINED A SIGNIFICANT NUMBER OF PAGES WHICH DO NOT REPRODUCE LEGIBLY.

294068

ADA015703

ON THE STABILITY OF NON-AUTONOMOUS DIFFERENTIAL EQUATIONS $\dot{x} = [A + B(t)]x$, WITH SKEW-SYMMETRIC MATRIX $B(t)$

by

A. P. Morgan and K. S. Narendra

August 1975

Control Theory

Becton Center Technical Report CT-66

DEPARTMENT OF ENGINEERING
AND APPLIED SCIENCE

YALE UNIVERSITY

OFFICE OF NAVAL RESEARCH

CONTRACT N00014-67-A-0097-0020

NR 375-131

ON THE STABILITY OF NON-AUTONOMOUS DIFFERENTIAL EQUATIONS $\dot{x} = [A + B(t)]x$, WITH SKEW-SYMMETRIC MATRIX $B(t)$

by

A. P. Morgan and K. S. Narendra

August 1975

The research reported in this document was sponsored in part by support extended to Yale University by the U. S. Office of Naval Research.

Technical Report No. CT-66

Becton Center

Yale University

New Haven, Connecticut

ON THE STABILITY OF NON-AUTONOMOUS

DIFFERENTIAL EQUATIONS $\dot{x} = [A + B(t)]x$,

WITH SKEW-SYMMETRIC MATRIX $B(t)$

A. P. Morgan and K. S. Narendra

ABSTRACT

In this paper we characterize (in Theorem 1) the uniform asymptotic stability of equations of the form

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \left[\begin{array}{c|c} A(t) & -B(t)^T \\ \hline B(t) & 0 \end{array}\right] \begin{bmatrix} x \\ y \end{bmatrix}$$

where $A + A^T$ is stable in terms of the "richness" of $B(t)$. The equation is stable if and only if $B(t)$ is sufficiently rich. We actually obtain stability results for a much broader class of systems (Theorems 2 and 3) whose behavior is similar to the one above. Such systems have come up recently in some adaptive control problems.

I. Introduction

The purpose of this paper is to characterize the uniform asymptotic stability of certain non-autonomous linear systems of the form

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \left[\begin{array}{c|c} A & -B^T \\ \hline C & 0 \end{array}\right] \begin{bmatrix} x \\ y \end{bmatrix}$$

where $A = A(t)$ is a time varying stable nxn matrix and $B = B(t)$, $C = C(t)$ are time varying mxn matrices. Such equations arise in connection with questions of adaptive identification and control as described in Narendra and Kudva [3].

Theorem 1 below is illustrative of the type of result we have obtained. It is a corollary to the more general Theorems 2 and 3. We state and discuss these results in section II giving examples and some indication of proofs, including the presentation of a key lemma.

Some results concerning (non-uniform) asymptotic stability have also been obtained, and these are stated in section III. In section IV we discuss in more detail the control applications of this work, which are summarized as Theorems 4 and 5. Section V contains the longer proofs.

Previous work on the stability properties of this type of system has been done by Yuan and Wonham [5]. They found sufficient conditions for asymptotic stability in the case that the system can be put in the form

$$\dot{e} = Ee + \Phi x + \Psi u$$

$$\dot{\Phi} = -\Gamma e x^T$$

$$\dot{\Psi} = -\Gamma e u \quad .$$

(See section IV, Theorem 4 for more details.) Anderson in [1] considered some almost periodic cases, obtaining sufficient conditions for uniform asymptotic stability.

II. Statement of Main Theorems

The following Theorem 1 gives a complete characterization of uniform asymptotic stability when $A + A^T$ is stable and $C = B$. It is a corollary to Theorems 2 and 3, which we will state after a discussion of Theorem 1.

First, however, we establish some notation and state several definitions. The nxn time varying matrix $A = A(t)$ is called "stable" if the system $\dot{x} = A(t)x$ is uniformly asymptotically stable. The length of $x \in R^n$ is denoted "$|x|$". If A is an nxn matrix, "$|A|$" denotes the uniform norm of A derived from $|x|$.

The equilibrium state $x \equiv 0$ of the uniformly stable differential equation $\dot{x} = f(x,t)$ is uniformly asymptotically stable (u.a.s.) if for some $\varepsilon_1 > 0$ and all $\varepsilon_2 > 0$ there is a $T = T(\varepsilon_1,\varepsilon_2) > 0$ such that if $x(t)$ is a solution and $|x(t_0)| \leq \varepsilon_1$, then $|x(t)| \leq \varepsilon_2$ for all $t \geq t_0 + T$. If T depends on t_0, then $x \equiv 0$ is (non-uniformly) asymptotically stable (a.s.).

<u>Theorem 1</u>. Let $A = A(t)$ be an nxn matrix of bounded piecewise continuous functions such that $A + A^T$ is stable. Let $B(t)$ be an mxn matrix of bounded piecewise continuous functions. Then the system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \left[\begin{array}{c|c} A & -B^T \\ \hline B & 0 \end{array}\right] \begin{bmatrix} x \\ y \end{bmatrix} \tag{1}$$

is u.a.s. if and only if there are positive numbers T_0, ε_0, and δ_0 such that given $t_1 \geq 0$ and a unit vector $w \in R^n$, there is a $t_2 \in [t_1, t_1 + T_0]$ such that

$$\left| \int_{t_2}^{t_2 + \delta_0} B(\tau)^T w \, d\tau \right| \geq \varepsilon_0 .$$

<u>Corollary 1.</u> If $B(t)$ is smooth, $|\dot{B}(t)|$ is uniformly bounded, and there are real numbers $a > 0$ and b such that

$$\int_{t_1}^{t_2} |B(\tau)^T w| d\tau \geq a(t_2 - t_1) + b$$

for all unit $w \in R^m$ and all $t_2 \geq t_1 \geq 0$, then (1) is u.a.s.

<u>Corollary 2.</u> If (1) is u.a.s., then there are real numbers $a > 0$ and b such that

$$\int_{t_1}^{t_2} |B(\tau)^T w| d\tau \geq a(t_2 - t_1) + b$$

for all unit vectors $w \in R^m$ and all $t_2 \geq t_1 \geq 0$.

The condition given in Theorem 1 is a "richness" condition for $B(t)$. It says that for any unit direction w, $B(t)^T w$ is "periodically" large; or, at least that $B(t)^T w$ does not damp down to zero. The condition requires that there be a fixed length of time, T_0, such that $B(t)$ "points in all directions" as t takes on values in any interval of length T_0. Also it requires that $B(t)$ maintain sufficient length. However, it requires even more than this, since the condition of corollary 2,

$$\int_{t_1}^{t_2} |B(\tau)^T w| d\tau \geq a(t_2 - t_1) + b \quad ,$$

is not sufficient.

It is therefore apparent that, for fixed w, the sign changes that $B(t)^T w$ goes through are also significant. $B(t)^T w$ must not only be periodically large (over intervals of length T_0) but it must maintain the same sign for a fixed interval of time (of length δ_0). (See the example below.)

It is immediate that the condition: there are positive numbers T_0 and ε_0 such that

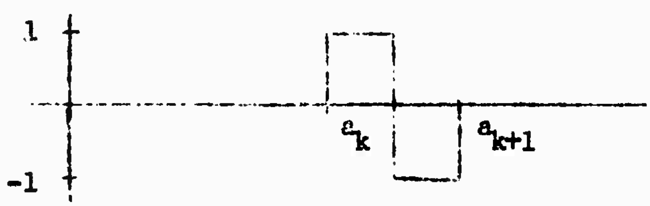
$$\left| \int_{t_1}^{t_2 + t_0} B(\tau)^T w \, d\tau \right| \geq \varepsilon_0$$

for all unit $w \in R^m$ and $t_2 \geq t_1 \geq 0$

is sufficient but not necessary for (1) to be u.a.s. (In this case δ_0 can be chosen arbitrarily and does not depend on w.)

The following example illustrates some of the above comments. Let $a_k = \sum_{n=1}^{k} \frac{1}{n}$, and define a square wave function with increasing frequency $u(t): [0,\infty) \to R^1$ by

$$u(t) = \begin{cases} 1 \text{ if } t \in [a_k, a_k + \frac{1}{2(n+1)}] \\ -1 \text{ if } t \in [a_k + \frac{1}{2(n+1)}, a_{k+1}] \end{cases}$$



then the two dimensional system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -\alpha & -u(t) \\ u(t) & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

(where α is a positive number)is not u.a.s. Note, however, that

$$\int_{t_1}^{t_2} |u(\tau)| d\tau = (t_2 - t_1).$$

Thus the necessary condition of corollary 2 is not sufficient. Also, compare this with the following comments on the category PS*.

We should point out that corollary 1 can be generalized as follows. Instead of requiring that B(t) be smooth and $|\dot{B}(t)|$ be bounded, we make the somewhat less restrictive assumption that the components of B(t) be contained in the set PS*, defined as a convenient category of input functions by Yuan and Wonham in [5]. (However, we must keep the integral

assumption.) PS* is the set of all piecewise smooth functions $g:[0,\infty) \to R^1$ that are uniformly bounded, whose derivatives are uniformly bounded (where defined), and for which the intervals over which g is smooth do not shrink to 0.

For example, an input function g defined to be constant on intervals (a_n, a_{n+1}) where $a_{n+1} - a_n$ is bounded below as $n \to \infty$ is in PS*.

Theorem 1 is an immediate corollary to the following two theorems, which are our main results.

Theorem 2. Let A(t) be a stable nxn matrix of bounded piecewise continuous functions. Let P(t) be a symmetric positive definite matrix of bounded continuous functions such that $\dot{P} + PA + A^TP$ is stable. (Many such P exist. See the discussion below.) Let B(t) be an nxm matrix of bounded piecewise continuous functions.

Assume that there exist positive numbers T_0, ε_0, and δ_0 such that given $t_1 \geq 0$ and a unit vector $w \in R^n$, there is a $t_2 \in [t_1, t_1 + T_0]$ such that

$$\left| \int_{t_2}^{t_2+\delta_0} B(\tau)^T w \, d\tau \right| \geq \varepsilon_0 .$$

Then the system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \left[\begin{array}{c|c} A & -B^T \\ \hline B \cdot P & 0 \end{array}\right] \begin{bmatrix} x \\ y \end{bmatrix} \tag{2}$$

is u.a.s.

Corollary 1 and the comments about Yuan and Wonham's PS* in the discussion following it hold exactly as written in this case.

"P(t) is positive definite" means that there exist positive constants α and β such that $\alpha x^Tx \leq x^TP(t)x \leq \beta x^Tx$ for all $x \in R^n$ and all t. We may interpret the stability of $\dot{P} + PA + A^TP$ to mean that $\dot{P} + PA + A^TP = -Q$ where $Q = Q(t)$ is positive definite.

By Krasovskii's theorem, the uniform asymptotic stability of $\dot{x} = A(t)x$ implies that given any continuous symmetric positive definite $Q(t)$, there exists a continuous symmetric positive definite $P(t)$ such that $\dot{P} + PA + A^T P = -Q$. (See Narendra and Taylor [4], p. 62, or Halanay [2], p. 44, theorem 1.6''. Note that the proof given in Halanay, although stated for continuous $A(t)$, holds for piecewise continuous $A(t)$.)

Theorem 3. Let $A(t)$ be a stable nxn matrix of bounded piecewise continuous functions. Let $B(t)$ and $C(t)$ be nxn matrices of bounded piecewise continuous functions. Suppose that the system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \left[\begin{array}{c|c} A & -B^T \\ \hline C & 0 \end{array}\right] \begin{bmatrix} x \\ y \end{bmatrix} \tag{3}$$

is u.a.s. Then there are positive real numbers T_0, δ_0, and ε_0 such that given $t_1 \geq 0$ and a unit vector w, there is a $t_2 \varepsilon [t_1, t_1 + T_0]$ such that

$$\left| \int_{t_2}^{t_2 + \delta_0} B(\tau)^T w \, d\tau \right| \geq \varepsilon_0 .$$

Corollary 2 holds exactly as written in this case also.

The comments made after the statement of Theorem 1 apply to Theorems 2 and 3. The condition which is necessary and sufficient for u.a.s. is a "richness" condition for $B(t)$, which however involves a subtlety concerning the sign changes of $B(t)^T w$ as $t \to \infty$.

The following is a key observation, used in the proof of Theorem 2. Consider equation (2), and assume that the hypothesis of Theorem 2 holds. We shall use the notation $z(t) = [x(t), y(t)]^T$ from now on.

Lemma. Let ε_1 and δ be given positive numbers. Then there is a $T = T(\varepsilon_1, \delta)$ such that if $z(t)$ is a solution of (2) and $|z(t_1)| \leq \varepsilon_1$, then there exists some $t_2 \varepsilon [t_1, t_1 + T]$ such that $|y(t_2)| \leq \delta$.

The lemma says that if $B(t)$ is sufficiently rich, then, for any solution $z(t) = [x(t), y(t)]^T$, $y(t)$ gets "periodically" small.

We will now outline the proof of Theorem 2, which is written out in detail in section V. Define the Lyapunov function $V(z,t) = V([x,y]^T,t) = x^T P(t)x + y^T y$. Then $\dot{V}(z,t) \leq -k|x|^2$ where k is some positive number. Thus if $|y|$ is small and $|z|$ is not, then $|x|$ is not. In this case, $|\dot{V}|$ is large and $V(z,t)$ is decreasing. Since $\dot{V} \leq 0$, we have uniform stability, and the observations of the previous two sentences show that $|z|$ is periodically getting smaller and smaller. Uniform asymptotic stability follows.

The proof of the lemma can be easily derived from the following two sublemmas.

<u>Sublemma 1.</u> Let $\varepsilon_1 > \varepsilon_2 > 0$. Then there is an $n = n(\varepsilon_1, \varepsilon_2)$ such that if $z(t) = [x(t), y(t)]^T$ is a solution of (2) with $|z(t_1)| \leq \varepsilon_1$ and $S = \{t \in [t_1, \infty) \,|\, |x(t)| > \varepsilon_2\}$, then $\mu(S) \leq n$ where μ denotes Lebesque measure.

This sublemma holds without any restriction on $B(t)$. It states that there is a uniform limit on the amount of time a solution starting inside the ε_1 ball can remain outside the ε_2 ball. It therefore implies that if $z(t)$ is any solution of (2), then $x(t) \to 0$. It also implies the following. Given $\varepsilon_1 > \varepsilon_2 > 0$, there is a $T > 0$ such that if $z(t)$ is a solution of (2) with $|z(t_1)| \leq \varepsilon_1$, then there is a $t_2 \in [t_1, t_1 + T]$ such that $|x(t_2)| \leq \varepsilon_2$.

We conjecture that sublemma 1 can be improved to say that $x(t) \to 0$ exponentially. (See discussion in section III.)

Assume that the hypothesis of Theorem 2 holds. Then we have

<u>Sublemma 2.</u> Let $\delta > 0$ and $\varepsilon_1 > 0$ be given. Then there exist positive numbers ε and T such that if $z(t)$ is a solution of (2) with $|z(t_1)| \leq \varepsilon_1$ and if $|y(t)| \geq \delta$ for $t \in [t_1, t_1 + T]$, then there is a $t_2 \in [t_1, t_1 + T]$ such that $|x(t_2)| \geq \varepsilon$.

Thus, if $b(t)$ is "rich" and $|y(t)|$ is large, then $|x(t)|$ must be periodically large. The lemma is established from the two sublemmas as follows. If $y(t)$ is not periodically small, then the sublemmas imply that $x(t)$ is periodically both large and small. But sublemma 1 puts an upper bound on this type of behavior. The details of the proof of the lemma are in section V.

III. Non-uniform Asymptotic Stability

We present three propositions concerning asymptotic stability, deferring proofs until section V.

<u>Proposition 1.</u> Let $A = A(t)$ be an nxn matrix of bounded piecewise continuous functions such that $A + A^T$ is stable. Let $B(t)$ be an mxn matrix of bounded piecewise continuous functions.

Assume that if $z(t) = [x(t), y(t)]^T$ is a solution to (1), then there are positive constants α and K such that

$$|x(t)| \leq K e^{-\alpha(t-t_0)} \quad \text{for all } t \geq t_0 .$$

(The choice of α and K may depend specifically on $z(t)$.)

Then (1) is asymptotically stable if and only if there are positive numbers ε_0 and δ_0 such that if $w \in R^m$ is a unit vector, then there is a sequence $t_n \to \infty$ with

$$\left| \int_{t_n}^{t_n+\delta_0} b(\tau)^T w \, d\tau \right| \geq \varepsilon_0 \text{ for all } n.$$

Proposition 1 is, of course, a non-uniform version of Theorem 1. Similar results analogous to Theorems 2 and 3 also hold. The condition requiring exponential convergence of $|x(t)|$ to zero can be weakened to the requirement that

$$\int_{t_0}^{\infty} |x(\tau)| d\tau < \infty .$$

We conjecture that this always happens (without any conditions on $B(t)$) and therefore that the exponential convergence condition can be omitted from Proposition 1. If this conjecture is true, then we would have a complete characterization of asymptotic stability analogous to that for uniform asymptotic stability.

Proposition 2 below shows that for the two dimensional case we can drop the exponential convergence condition from the sufficiency part of Proposition 1. However, in the proof we avoid, rather than settle, the conjecture.

<u>Proposition 2.</u> Let $a:[t_0,\infty) \to R^1$ be bounded, piecewise continuous and stable. Let $b:[t_0,\infty) \to R^1$ be piecewise continuous and bounded. If there are positive constants ε_0 and δ_0 and a sequence $t_n \to \infty$ such that

$$\left| \int_{t_n}^{t_n+\delta_0} b(\tau) d\tau \right| \geq \varepsilon_0 \quad \text{for all } n,$$

then the two dimensional system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a(t) & -b(t) \\ b(t) & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{4}$$

is asymptotically stable.

The following proposition is used to prove both Proposition 1 and 2.

<u>Proposition 3.</u> Let A and B be as given in Theorem 2. Consider solutions $z(t)$ to equation (2). If, for every $z(t)$ and $\delta > 0$, there is a sequence $t_n \to \infty$ such that $|y(t_n)| \leq \delta$, then (2) is asymptotically stable.

This result says that if the non-uniform analog to the lemma for Theorem 2 holds, then the non-uniform analog for Theorem 2 holds.

IV. Applications to Control Theory

The type of equations discussed in earlier sections have come up recently in connection with control problems dealing with the adaptive observer. It also appears reasonable to assume that questions regarding the uniform asymptotic stability of similar non-autonomous equations will increasingly occur in adaptive control problems where parameters of the systems can be adjusted at the discretion of the designer, i.e. parts of the vector differential equation can be chosen. In this section we characterize the uniform asymptotic stability of two types of equations which arose in the context of identification (See Narendra and Kudva [3], for details. Also compare Yuan and Wonham [5]).

<u>Theorem 4.</u> Consider the system

$$\begin{aligned} \dot{e} &= Ee + \Phi x + \Psi u \\ \dot{\Phi} &= -\Gamma e x^T \\ \dot{\Psi} &= -\Gamma e u^T \end{aligned} \tag{5}$$

where E is a stable $n\times n$ constant matrix, $e \in R^n$, Φ is $n\times n$, Ψ is $n\times m$, Γ is a symmetric positive definite matrix such that $\Gamma E + E^T\Gamma$ is stable, and $x:[t_0,\infty) \to R^n$, $u:[t_0,\infty) \to R^m$ are piecewise continuous, uniformly bounded vector valued functions.

Then (5) is u.a.s. if and only if there are positive constants $T_0, \varepsilon_0, \delta_0$ such that given $t_1 \geq 0$ and a unit vector

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \varepsilon\, R^n \times R^m = R^{n+m} ,$$

there is a $t_2 \in [t_1, t_1 + T_0]$ such that

$$\left| \int_{t_2}^{t_2+\delta_0} [x(\tau)^T, u(\tau)^T] w \, d\tau \right| \geq \varepsilon_0 .$$

If $|\dot{x}(t)|$ and $|\dot{u}(t)|$ are defined and bounded, then the above condition can be replaced by:

There exist $a > 0$ and b such that

$$\int_{t_1}^{t_2} |[x(\tau)^T, u(\tau)^T] w| \, d\tau \geq a(t_2 - t_1) + b$$

for all unit $w \in R^n \times R^m$ and all $t_2 \geq t_1$.

This completes the statement of Theorem 4.

In the context of identification, $\dot{x} = Ax + Bu$ where A is a constant stable matrix and B is a constant matrix. Thus $\dot{x}$ is always bounded.

This theorem follows at once from Theorems 2 and 3 and their corollaries. Note also the comments in section II which allow us to assume "$u(t) \in PS^*$" in place of "$|\dot{u}(t)|$ bounded".

The next theorem concerns a type of equation which also arises in identification schemes. (See Narendra and Kudva [3], p. 553. Also see Anderson [1], p. 2.20.)

<u>Theorem 5.</u> Let A be a stable $n\times n$ constant matrix, and let P be a positive definite symmetric matrix such that $PA + A^TP$ is stable. Assume that there exist non-zero vectors d and h such that $Pd = h$. Let $v(t)$ be a piecewise continuous bounded vector valued function. Then the system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \left[\begin{array}{c|c} A & h \cdot v(t)^T \\ \hline -v(t) \cdot d^T & 0 \end{array}\right] \begin{bmatrix} x \\ y \end{bmatrix}$$

is u.a.s. if and only if there are positive constants T_0, ε_0, and δ_0 such that if $t_1 \geq 0$ and w is a unit vector, then there is a $t_2 \in [t_1, t_1 + T_0]$ such that

$$\left| \int_{t_2}^{t_2+\delta_0} v(\tau)^T \cdot w \, d\tau \right| \geq \varepsilon_0 .$$

If $|\dot{v}(t)|$ is bounded or $v(t) \in PS^*$, then the above condition can be replaced by.

there exist $a > 0$ and b such that

$$\int_{t_1}^{t_2} |v(\tau)^T \cdot w| d\tau \geq a(t_2 - t_1) + b \quad \text{for all unit } w.$$

This completes the statement of Theorem 5.

This theorem is immediate from Theorems 2,3, and corollaries. (Of course, we could also easily derive a version of Theorem 5 with A,h,d, and P time varying. However, these are constant in the application cited.)

Note that, by the Kalman-Yakubovich lemma, the conditions

a) $PA + A^TP$ stable for some positive definite symmetric P, and

b) $Pd = h$ for some d,h

are equivalent to the condition that the transfer matrix $H(s) \equiv h^T(sI-A)^{-1}d$ be positive real. (Narendra and Taylor [4], p. 49.)

V. Proofs of Theorems

We shall present proofs to Theorem 2, the lemma and sublemmas, Theorem 3, and Propositions 1,2, and 3. The proofs of the corollaries and the comment in section II about PS* are routine and therefore omitted.

It will be convenient to use the notation

$$z = \begin{bmatrix} x \\ y \end{bmatrix} = [x,y]^T \in R^{n+m} .$$

Also, for convenience, we assume $|A(t)| \leq 1$ and $|B(t)| \leq 1$ for all t.

Proof of Theorem 2.

1. By hypothesis we may choose positive constants α,β,a,b such that

$$\alpha\, x^Tx \leq x^T P(t)x \leq \beta x^T x \quad \text{and}$$

$$a\, x^Tx \leq x^T Q(t)x \leq bx^Tx \quad \text{for all } x,$$

where $-Q(t) \equiv \dot{P}(t) + P(t)A(t) + A(t)^T P(t)$. Without loss of generality, assume $\beta \geq 1$ and $a \leq 1$.

Define $V(z,t) = x^T P(t)x + y^T y$. Then

$\dot{V}(z,t) = 2x^T(\dot{P}(t) + P(t)A(t) + A(t)^T P(t))x = -2x^T Q(t)x \leq -2ax^T x$.

If $z(t)$ is a solution of (2), then the above implies that $|z(t)|$ is monotonically non-increasing as $t \to \infty$. Thus we have uniform stability.

2. We will now show that given $\varepsilon_1 > \varepsilon_2 > 0$ there is a γ with $0 < \gamma < 1$ and an $M > 0$ such that if $z(t)$ is a solution of (2) with

$$\varepsilon_2 \leq V(z(t),t) \leq \varepsilon_2 \quad \text{for} \quad t\,\varepsilon[t_1,t_1 + M],$$

then there is a $t_2\, \varepsilon[t_1,t_1 + M]$ such that $V(z(t_2),t_2) \leq \gamma \cdot V(z(t_1),t_1)$.

Since $V(z(t),t)$ is non-increasing, this implies uniform asymptotic stability. The above fact follows routinely from the lemma and the relation $\dot{V}(z(t),t) \leq -2ax(t)^T x(t)$. However, for completeness, we will write out the details.

Choose positive numbers c_1 and c_2 so that $1-c_1 > 0$, $\sqrt{(1-c_1)}/\sqrt{\beta} - 2c_2/\sqrt{\alpha} > 0$ and $0 < 2ac_2(\sqrt{(1-c_1)}/\sqrt{\beta} - 2c_2/\sqrt{\alpha})^2 < 1$.

(Say $c_1 = 3/4$ and $c_2 = \sqrt{\alpha}\,/8\sqrt{\beta}$.)

Use the lemma to obtain T when $\varepsilon = \varepsilon_1$ and $\delta = \varepsilon_2 \cdot c_1$.

Define $\gamma = 1 - [2ac_2(\sqrt{(1-c_1)}/\sqrt{\beta} - 2c_2/\sqrt{\alpha})^2]$ and $M = T + c_2$. We shall show that for this γ and M our result holds.

We want $0 < \gamma < 1$. But this is clear from the choice of c_1 and c_2. Let $t_2'\, \varepsilon[t_1,t_1 + T]$ be such that $|y(t_2')| \leq \delta = \varepsilon_2 \cdot c_1$. If $V(z(t_2'),t_2') \leq \varepsilon_2$, we are done. Assume $V(z(t_2'),t_2') \geq \varepsilon_2$. Then

$$V(z(t_2'),t_2') = x(t_2')^T\, P(t_2')x(t_2') + |y(t_2')|^2 \text{ implies}$$

$$\beta|x(t_2')|^2 \geq V(z(t_2'),t_2') - \delta \geq V(z(t_2'),t_2')(1-c_1).$$

Now $\dot{x} = Ax - B^T y$ gives, for any $t \geq t_2'$,

$$|x(t_2')| - |x(t)| \leq |x(t) - x(t_2')| \leq \int_{t_2'}^{t} |A(\tau)x(\tau) - B(\tau)y(\tau)|d\tau$$

$$\leq (1 + 1)|z(t_2')| \ (t-t_2') = 2|z(t_2')| \ (t-t_2'),$$

since we have assumed $|A(\tau)| \leq 1$ and $|B(\tau)| \leq 1$ for all τ.

If we let $t_2 = t_2' + c_2$, then we see that

$$\begin{aligned} |x(t)| &\geq |x(t_2')| - 2(t_2 - t_2')|z(t_2')| \\ &\geq (\sqrt{(1-c_1)}/\sqrt{\beta}) \ \sqrt{V(z(t_2'),t_2')} - 2c_2|z(t_2')| \\ &\geq (\sqrt{(1-c_1)}/\sqrt{\beta}) \ \sqrt{V(z(t_2'),t_2')} - (2c_2/\sqrt{\alpha}) \ \sqrt{V(z(t_2'),t_2')} \\ &\geq \sqrt{V(z(t_2'),t_2')} \ (\sqrt{(1-c_1)}/\sqrt{\beta} - 2c_2/\sqrt{\alpha}) \end{aligned}$$

for all $t \in [t_2',t_2]$.

Then $V(z(t_2'),t_2') - V(z(t_2),t_2) =$

$$\int_{t_2'}^{t_2} -\dot{V}(z(\tau),\tau)d\tau \geq 2a \int_{t_2'}^{t_2} |x(\tau)|^2 d\tau \geq$$

$$2a \cdot c_2 \cdot V(z(t_2'),t_2') \cdot (\sqrt{(1-c_1)}/\sqrt{\beta} - 2c_2/\sqrt{\alpha})^2 .$$

Thus $V(z(t_2),t_2) \leq V(z(t_2'),t_2') \cdot \gamma$, and we are done.

<u>Proof of the Lemma.</u>

Let $\delta > 0$. By the comments after the statement of sublemma 1 and by sublemma 2, the assumption for some solution $z(t)$ that $|y(t)| \geq \delta$ implies that there is an $\varepsilon > 0$ such that $|x(t)|$ is repeatedly both less than $\varepsilon/2$ and greater than ε. Now this eventually leads to a contradiction with sublemma 1, when we let $\varepsilon_1 = |z(t_1)|$ and $\varepsilon_2 = \varepsilon/2$. Since all these results are uniform, we conclude that $|y(t)| \leq \delta$ repeatedly (uniformly). This yields the lemma.

Proof of Sublemma 1.

This is immediate from the relation $\dot{V}(z,t) \leq -2ax^Tx$. We can choose $n(\varepsilon_1,\varepsilon_2) = \varepsilon_1^2/2a\varepsilon_2^2$.

Proof of Sublemma 2.

1. By hypothesis, we have T_0, ε_0, and δ_0 given, obeying the condition in the statement of Theorem 2. Let $z(t)$ be a solution with initial condition $|z(t_1)| \leq \varepsilon_1$. Suppose that $|y(t)| \geq \delta$ for all $t \in [t_1, t_1 + T]$ where $T \equiv T_0 + \delta_0$.

2. Now $\dot{x} = Ax - B^Ty$ implies, for any $t \geq t_1$, that

$$x(t + \delta_0) = x(t) + \int_t^{t+\delta_0} A(\tau)x(\tau) - B(\tau)^T y(\tau)d\tau, \text{ which gives}$$

$$|x(t + \delta_0)| \geq \left| \int_t^{t+\delta_0} B(\tau)^T y(\tau)d\tau \right| - \left| x(t) + \int_t^{t+\delta_0} A(\tau)x(\tau)d\tau \right| .$$

We shall see below that we can make the second term arbitrarily small and the first term relatively large by appropriate choices of t and ε. This will prove the result.

3. We have $\dot{y}(\tau) = B(\tau)P(\tau)x(\tau)$. Thus, "when x is small, y is flat." More precisely, given T' and M' positive constants, there is a $\theta > 0$ such that if $z(\tau) = [x(\tau), y(\tau)]^T$ is a solution to (2) with $|x(\tau)| \leq \theta$ for all $\tau \in [t_1, t_1 + T']$, then $|y(\tau) - y(t_1)| \leq \varepsilon'$ for all $\tau \in [t_1, t_1 + T]$.

Let $\varepsilon' = \dfrac{\varepsilon_0 \cdot \delta}{2\delta_0}$, $M' = \varepsilon_1$, and $T' = T = T_0 + \delta_0$, and fix θ for these choices.

4. Define $\varepsilon = \min \{\dfrac{\delta\varepsilon_0}{8}, \dfrac{\delta\varepsilon_0}{8\delta_0}, \theta\}$. We shall now show that the sublemma holds for this choice of T and ε. If $|x(t_2)| \geq \varepsilon$ for some $t_2 \in [t_1, t_1 + T]$, we are done. Assume $|x(t)| \leq \varepsilon$ for all $t \in [t_1, t_1 + T]$.

Then $\left|x(t) + \int_t^{t+\delta_0} A(\tau)x(\tau)d\tau\right| \leq \varepsilon + \varepsilon \cdot \delta_0 \leq \dfrac{\varepsilon_0\delta}{4}$ for any $t \in [t_1, t_1 + T_0]$. (We have assumed $|A(\tau)| \leq 1$ and $|B(\tau)| \leq 1$ for all τ.)

By hypothesis there is a $t' \in [t_1, t_1 + T_0]$ such that $\left| \int_{t'}^{t'+\delta_0} B(\tau)^T w d\tau \right| \geq \varepsilon_0$ where $w \equiv \dfrac{y(t_1)}{|y(t_1)|}$.

But

$$\left| \int_{t'}^{t'+\delta_0} B(\tau)^T (w\ |y(t_1)| - y(\tau)) d\tau \right| \leq \int_{t'}^{t'+\delta_0} |y(t_1) - y(\tau)| d\tau$$

$$\leq \delta_0 \cdot \frac{\varepsilon_0 \delta}{2\delta_0} = \frac{\varepsilon_0 \delta}{2} ,$$

because $|x(\tau)| \leq \varepsilon \leq \theta$ for $\tau \in [t_1, t_1 + T]$. (See 3. above.)

Therefore

$$|y(t_1)| \left| \int_{t'}^{t'+\delta_0} B(\tau)^T w\, ds \right| - \left| \int_{t'}^{t'+\delta_0} B(\tau)^T y(\tau) d\tau \right| \leq \frac{\varepsilon_0 \delta}{2} ,$$

implying $\left| \int_{t'}^{t'+\delta_0} B(\tau)^T y(\tau) dz \right| \geq \varepsilon_0 \delta - \dfrac{\varepsilon_0 \delta}{2} = \dfrac{\varepsilon_0 \delta}{2}$.

Thus $|x(t' + \delta_0)| \geq \dfrac{\varepsilon_0 \delta}{2} - \dfrac{\varepsilon_0 \delta}{4} = \dfrac{\varepsilon_0 \delta}{4} > \varepsilon$.

This completes the proof of sublemma 2.

<u>Proof of Theorem 3.</u>

1. Assume (to get a contradiction) that the conclusion of the theorem is false. Then given any positive $T_0, \delta_0, \varepsilon_0$, there is a unit vector w and a $t_1 \geq 0$ such that

$$\left| \int_{t_2}^{t_2+\delta_0} B(\tau)^T w\, d\tau \right| \leq \varepsilon_0 \quad \text{for each } t_2 \in [t_1, t_1 + T_0].$$

2. Since (3) is assumed to be u.a.s., there is a T such that if $z(t)$ is a solution with $|z(t_1)| \leq 1$, then $|z(t_1 + T| \leq \frac{1}{2}$. Fix this T for the remainder of the proof.

3. Define

$$D(t) = \left[\begin{array}{c|c} A(t) & 0 \\ \hline C(t) & 0 \end{array}\right] \quad \text{and} \quad E(t) = \left[\begin{array}{c|c} 0 & -B(t)^T \\ \hline 0 & 0 \end{array}\right]$$

and compare $\dot{z} = [D + E]z$ and $\dot{z}' = Dz'$ via variation of constants:

$$z(t) = z'(t) + \int_{t_1}^{t} \Phi'(t,\tau)\, E(\tau)z(\tau)d\tau$$

where $z(t_1) = z'(t_1)$ and $\Phi'(t,\tau)$ is the state transition matrix of $\dot{z}' = D(t)z'$.

The proof now proceeds as follows. We will find a unit vector w and a $t_1 \geq 0$ such that the solution z(t) to (3) with $z(t_1) = [0,w]^T$ obeys

$$\left| \int_{t_1}^{t_1+T} \Phi'(t_1+T,\tau)\, E(\tau)z(\tau)d\tau \right| \leq \frac{3}{8} \quad . \quad (*)$$

This, combined with the variation of constants formula, implies that $|z(t_1)| = 1$ and $|z(t_1+T)| > \frac{1}{2}$, contradicting the choice of T. (Note that $z'(t) = [0,w]^T$ is a constant solution for $\dot{z}' = Dz'$.)

3. We will need the following, which is easy to prove.

(i) There is a constant $R > 0$ such that $|\Phi'(t,\tau)| \leq R$ whenever $t \geq \tau \geq 0$.

(ii) Given $\varepsilon > 0$, there is a $\delta > 0$ such that if $|\tau_1 - \tau_2| \leq \delta$ and $t \geq \tau_i$ for $i = 1,2$, then $|\Phi'(t,\tau_1) - \Phi'(t,\tau_2)| \leq \varepsilon$.

Let $\varepsilon = \frac{1}{16T}$ and choose δ so that (ii) holds. We lose no generality assuming $\delta \leq T$. Fix this δ for the remainder of the proof.

4. We will establish (*) in 2. above as follows. First we show that we can "factor $\Phi'(t_1 + T,\tau)$ out of the integral". This will be done by partitioning the interval $[t_1,t_1 + T]$ small enough via $t_1 < t_2 < \ldots < t_r = t_1 + T$ so that $\Phi'(t_1 + T,\tau)$ is essentially equal to the constant matrix $\Phi'(t_1 + T, t_i)$ for $\tau \in [t_{i-1},t_i]$. Then, by carefully choosing t_1 and w, we make

$$\left| \int_{t_1}^{t_1+T} E(\tau)z(\tau)d\tau \right| = \left| \int_{t_1}^{t_1+T} B(\tau)y(\tau)d\tau \right| \quad \text{small.}$$

It is important here that we can use the u.a.s. of (3) and the fact that $\dot{y} = C(t)x$ to conclude that y(t) becomes arbitrarily "flat" uniformly as $t \to \infty$. The result then follows.

5. Assume t_1 has been chosen. (We shall specify how to do this later.) Partition $[t_1,t_1 + T]$ using $t_1 < t_2 < \ldots < t_r$ where $t_{i+1} - t_i \leq \delta$ and $r \leq \frac{T}{\delta} + 2$.

(Recall that T is defined in 2. and δ in 3. above.) Then

$$\left| \int_{t_1}^{t_1+T} \Phi'(t_1 + T,\tau)\, E(\tau)z(\tau)d\tau - \sum_{i=2}^{r} \int_{t_{i-1}}^{t_i} \Phi'(t_1 + T,t_i)\, E(\tau)z(\tau)d\tau \right|$$

$$\leq \sum_{i=2}^{r} \int_{t_{i-1}}^{t_i} |\Phi'(t_1 + T,\tau) - \Phi'(t_1 + T,t_i)|\; |E(\tau)||z(\tau)|d\tau$$

$$\leq \sum_{i=2}^{r} \delta \cdot \frac{1}{16T} \cdot 1\cdot 1 \leq \frac{1}{8} \;. \quad \text{Also} \quad \left| \sum_{i=2}^{r} \int_{t_{i-1}}^{t_i} \Phi'(t_1 + T,t_i)\, E(\tau)z(\tau)d\tau \right| \leq$$

$$\sum_{i=2}^{r} R \left| \int_{t_{i-1}}^{t_i} B(\tau)^T y(\tau)d\tau \right| \;.$$

We shall show that with an appropriate choice of t_1 and $w = y(t_1)$, we have

$$\left| \int_{t_{i-1}}^{t_i} B(\tau)^T y(\tau)d\tau \right| \leq \frac{\delta}{8RT} \quad \text{for all } i \;.$$

Then the last inequality above will be bounded by $\frac{1}{4}$. Combining this with the previous inequality yields (*) in 2.

6. We now show how to choose t_1 and w.

Since $\dot{y} = Cx$ and (3) is u.a.s., we may show that there is a $t' > 0$ such that if z(t) is a solution with $|z(t_1)| \leq 1$ and $t_1 \geq t'$, then $|y(\tau) - y(t_1)| \leq \frac{1}{RT16}$ for all $\tau \in [t_1,t_1 + T]$. (Compare 4. above and the proof of sublemma 2, part 3.) Thus

$$\left| \int_{t_{i-1}}^{t_i} B(\tau)^T y(\tau)d\tau - \int_{t_{i-1}}^{t_i} B(\tau)^T y(t_1)d\tau \right| \leq \int_{t_{i-1}}^{t_i} |B(\tau)||y(\tau) - y(t_1)|\, d\tau$$

$$\leq \delta \cdot \frac{1}{RT16} \quad \text{when } t_1 + T \geq t_i \geq t_{i-1} \geq t_1 \geq t' \text{ and } t_i - t_{i-1} \leq \delta.$$

We now apply 1. above. Let $(T_0,\delta_0,\varepsilon_0) = (T + t',\delta,\delta/RT16)$, and conclude that there is a unit vector w and a $t_1 \geq t'$ such that

$$\left| \int_{t_{i-1}}^{t_i} B(\tau)^T w\, d\tau \right| \leq \frac{\delta}{RT16}$$

for all $t_1 + T \geq t_i \geq t_{i-1} \geq t_1$ when $t_i - t_{i-1} \leq \delta$.

It follows that

$$\left|\int_{t_{i-1}}^{t_i} B(\tau)^T y(\tau)d\tau\right| \leq \frac{\delta}{RT8} \quad \text{for all } i,$$

using $y(t_1) = w$ and the above two inequalities. This completes the proof of Theorem 3.

Proof of Proposition 1.

First note that $|\dot{y}| = |Cx| \leq |x| \leq |z_0|Ke^{-\alpha(t-t_0)}$ implies $\int_{t_0}^{\infty} |\dot{y}(t)|dt < \infty$. Therefore, given $\varepsilon > 0$, we can find $t_1 \geq t_0$ such that $|y(t) - y(t_1)| \leq \varepsilon$ for all $t \geq t_1$. Now a non-uniform version of sublemma 2 holds, and this implies the following non-uniform version of the lemma:

Lemma. Given $\delta > 0$, there is a sequence $t_n \to \infty$ such that $|y(t_n)| \leq \delta$.

With this lemma, sufficiency follows from Proposition 3.

The proof of Theorem 3 is also easily adapted to establish necessity, using the above "flatness" of $y(t)$.

Proof of Proposition 2.

We use Proposition 3. The condition on y can be established by a simple adaptation of the proof of the lemma. Unfortunately, this adaptation does not seem to generalize to higher dimensions.

Proof of Proposition 3.

This is exactly like the proof of Theorem 2 from the lemma.

References:

1. Anderson, B., "Multivariable Adaptive Identification," June 1974 (Preprint: Department of Electrical Engineering, University of Newcastle, New South Wales, 2308, Australia).

2. Halanay, A., Differential Equations, Academic Press, 1966.

3. Narendra, K. S. and P. Kudva, "Stable Adaptive Schemes for System Identification and Control, Parts I and II," IEEE Transactions on Systems, Man and Cybernetics, November 1974, pp. 542-560.

4. Narendra, K. S. and J. Taylor, Frequency Domain Criteria for Absolute Stability, Academic Press, 1973.

5. Yuan, J.S.C. and W. M. Wonham, "Asymptotic Identification Using Stability Criteria, Part I: Probing Signal Description," November 1974, Control Systems Report No. 7422, University of Toronto, Toronto, Canada.